**Introduction**

Tensile properties are used in selecting materials for different applications. Material specifications often include minimum tensile properties to ensure quality so tests must be made to guarantee that materials meet these specifications. Tensile properties are also used in research and development to compare new materials or processes. With plasticity theory (Chapter 5), tensile data can be used to predict a material’s behavior under forms of loading other than uniaxial tension.

Often the primary concern is strength. The level of stress that causes appreciable plastic deformation is called its *yield stress*. The maximum tensile stress that a material carries is called its *tensile strength* (or *ultimate strength* or *ultimate tensile strength*). Both measures are used, with appropriate caution, in engineering design. A material’s ductility may also be of interest. Ductility describes how much the material can deform before it fractures. Rarely, if ever, is the ductility incorporated directly into design. Rather, it is included in specifications to ensure quality and toughness. Elastic properties may be of interest, but these are measured ultrasonically.

**Tensile Specimens**

*Figure 3.1* shows a typical tensile specimen. It has enlarged ends or shoulders for gripping. The important part of the specimen is the gauge section. The cross-sectional area of the gauge section is less than that of the shoulders and grip region, so the deformation will occur here. The gauge section should be long compared to the diameter (typically, four times). The transition between the gauge section and

![Diagram of tensile specimen](image)

*Figure 3.1. Typical tensile specimen with a reduced gauge section and larger shoulders.*
Figure 3.2. Systems for gripping tensile specimens. For round specimens, these include threaded grips (a), serrated wedges (b), and split collars for butt-end specimens (c). Sheet specimens may be gripped by pins (d) or serrated wedges (e). From W. F. Hosford in *Tensile Testing*, ASM Int. (1992).

the shoulders should be gradual to prevent the larger ends from constraining deformation in the gauge section.

There are several ways of gripping specimens, as shown in Figure 3.2. The ends may be screwed into threaded grips, pinned, or held between wedges. Special grips are used for specimens with butt ends. The gripping system should ensure that the slippage and deformation in the grip region are minimized. It should also prevent bending.

**Stress–Strain Curves**

Figure 3.3 is a typical engineering stress–strain curve for a ductile material. For small strains, the deformation is elastic and reverses if the load is removed. At higher stresses, plastic deformation occurs. This is not recovered when the load is removed. Bending a wire or paper clip with the fingers (Figure 3.4) illustrates the difference. For small strains, the deformation is elastic and reverses if the load is removed. At higher stresses, plastic deformation occurs. This is not recovered when the load is removed. If the wire is bent, a small amount of it will snap back when released. However, if the bend is more severe, it will only partly recover, leaving a permanent bend. The onset of plastic deformation is usually associated with the first deviation of the stress–strain curve from linearity.*

* For some materials, there may be nonlinear elastic deformation.
It is tempting to define an *elastic limit* as the stress that causes the first plastic deformation and to define a *proportional limit* as the first departure from linearity. However, neither definition is very useful because they both depend on how accurately strain is measured. The more accurate the strain measurement is, the lower is the stress at which plastic deformation and nonlinearity can be detected.

To avoid this problem, the onset of plasticity is usually described by an *offset yield strength*. It is found by constructing a straight line parallel to the initial linear portion of the stress–strain curve, but offset from it by $e = 0.002$ (0.2%). The offset yield strength is taken as the stress level at which this straight line intersects the stress–strain curve (Figure 3.5). The rationale is that if the material had been loaded to this stress and then unloaded, the unloading path would have been along this offset line, resulting in a plastic strain of $e = 0.002$ (0.2%). The advantage of this way of defining yielding is that it is easily reproduced. Occasionally, other offset strains are used, and yielding is defined in terms of the stress necessary to achieve a specified total strain (e.g., 0.5%) instead of a specified plastic strain. In all cases, the criterion should be made clear to the user of the data.

Figure 3.3. Typical engineering stress–strain curve for a ductile material.

Figure 3.4. Using the fingers to sense the elastic and plastic responses of a wire. With a small force (*top*), all bending is elastic and disappears when the force is released. With a greater force (*bottom*), the elastic part of the bending is recoverable, but the plastic part is not. From W. F. Hosford, *Ibid.*
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Figure 3.5. Low strain region of the stress–strain curve for a ductile material. From W. F. Hosford, *Ibid*.

*Yield points:* The stress–strain curves of some materials (e.g., low-carbon steels and linear polymers) have an initial maximum followed by lower stress, as shown in Figures 3.6a and 3.6b. At any given instant after the initial maximum, the deformation occurs within a relatively small region of the specimen. For steels, this deforming region is called a *Lüder’s band.* Continued elongation occurs by movement of the Lüder’s band along the gauge section, rather than by continued deformation within the band. Only after the band has traversed the entire gauge section, does the stress rise again. In the case of linear polymers, the yield strength is usually defined as the initial maximum stress. For steels, the subsequent lower yield strength is used to describe yielding. Because the initial maximum stress is extremely sensitive to the alignment of the specimen, it is not useful in describing yielding. Even so,

Figure 3.6. Inhomogeneous yielding of low-carbon steel (*left*) and a linear polymer (*right*). After the initial stress maximum, the deformation in both materials occurs within a narrow band that propagates the length of the gauge section before the stress rises again.
the lower yield strength is sensitive to the strain rate. ASTM standards should be followed. The stress level during Lüder’s band propagation fluctuates. Some laboratories report the minimum level as the yield strength, and others use the average level.

As long as the engineering stress–strain curve rises, the deformation will occur uniformly along the gauge length. For a ductile material, the stress will reach a maximum well before fracture. When the maximum is reached, the deformation localizes forming a neck, as shown in Figure 3.7.

The tensile strength (or ultimate strength) is defined as the highest value of the engineering stress (Figure 3.8). For ductile materials, the tensile strength corresponds to the point at which necking starts. Less ductile materials fracture before they neck. In this case, the fracture stress is the tensile strength. Very brittle materials (e.g., glass) fracture before they yield. Such materials have tensile strengths, but no yield stresses.

**Ductility**

Two common parameters are used to describe the ductility of a material. One is the percent elongation, which is simply defined as

\[
\% El = \left( \frac{L_f - L_o}{L_o} \right) \times 100\%,
\]

where \(L_o\) is the initial gauge length and \(L_f\) is the length of the gauge section at fracture. Measurements may be made on the broken pieces or under load. For most
materials, the elastic elongation is so small compared to the plastic elongation that it can be neglected. When this is not so, as with brittle materials or with rubberlike materials, it should be made clear whether or not the percent elongation includes the elastic portion.

The other common measure of ductility is the percent reduction of area at fracture, defined as

\[
\% RA = \frac{(A_o - A_f)}{A_o} \times 100\%,
\]

where \(A_o\) is the initial cross-sectional area and \(A_f\) is the cross-sectional area of the fracture. If the failure occurs before the necking, the \(\% EI\) can be calculated from the \(\% RA\) by assuming constant volume. In this case,

\[
\% EI = \frac{\% RA}{(100 - \% RA)} \times 100\%.
\]

The \(\% EI\) and \(\% RA\) are no longer directly related after a neck has formed.

Percent elongation, as a measure of ductility, has the disadvantage that it combines the uniform elongation that occurs before necking and the localized elongation that occurs during necking. The uniform elongation depends on how the material strain hardens rather than on the fracture behavior. The necking elongation is sensitive to the specimen shape. With a gauge section that is very long compared to the diameter, the contribution of necking to the total elongation is very small. On the other hand, if the gauge section is very short, the necking elongation accounts for most of the elongation. For round bars, this problem has been remedied by standardizing the ratio of the gauge length to diameter at 4 : 1. However, there is no simple relation between the percent elongation of such standardized round bars and the percent elongation measured on sheet specimens or wires.

Percent reduction of area, as a measure of ductility, does not depend on the ratio of the gauge length to diameter. However, for very ductile materials, it is difficult to measure the final cross-sectional area, especially with sheet specimens.

**True Stress and Strain**

If the tensile tests are used to predict how the material will behave under other forms of loading, true stress–true strain curves are useful. The true stress, \(\sigma\), is defined as

\[
\sigma = \frac{F}{A},
\]

where \(A\) is the instantaneous cross-sectional area corresponding to the force, \(F\). Before necking begins, the true strain, \(\varepsilon\), is given by

\[
\varepsilon = \ln\left(\frac{L}{L_o}\right).
\]

The engineering stress, \(s\), is defined as the force divided by the original area, \(s = \frac{F}{A_o}\), and the engineering strain, \(e\), as the change in length divided by the original length, \(e = \frac{\Delta L}{L_o}\). As long as the deformation is uniform along the gauge length,
the true stress and true strain can be calculated from the engineering quantities. With constant volume, $LA = L_o A_o$,

$$A_o / A = L / L_o,$$  \hfill (3.6)

so $A_o / A = 1 + e$. Rewriting equation (3.4) as $\sigma = (F/A_o)(A_o/A)$ and substituting $A_o / A = 1 + e$ and $s = F/A_o$,

$$\sigma = s(1 + e).$$  \hfill (3.7)

Substitution of $L/L_o = 1 + e$ into equation (3.5) gives

$$\varepsilon = \ln(1 + e).$$  \hfill (3.8)

These expressions are valid only if the deformation is uniformly distributed along the gauge section. After necking starts, equation (3.4) is still valid for true stress, but the cross-sectional area at the base of the neck must be measured independently. Equation (3.5) could still be used if $L$ and $L_o$ were known for a gauge section centered on the middle of the neck and so short that the variations of area along its length are negligible. Then equation (3.6) would be valid over such a gauge section, so the true strain can be calculated as

$$\varepsilon = \ln(A_o/A),$$  \hfill (3.9)

where $A$ is the area at the base of the neck. Figure 3.9 shows the comparison of engineering and true stress–strain curve for the same material.

**EXAMPLE PROBLEM 3.1:** In a tensile test, a material fractured before necking. The true stress and strain at fracture were 630 MPa and 0.18, respectively. What is the tensile strength of the material?

**Solution:** The engineering strain at fracture was $e = \exp(0.18) - 1 = 0.197$. Because $s = \sigma / (1 + e)$, the tensile strength $= 630/1.197 = 526$ MPa.

![Figure 3.9](image-url) Figure 3.9. Comparison of engineering and true stress–strain curves. Before necking, a point on the true stress–strain curve ($\sigma-\varepsilon$) can be constructed from a point on the engineering stress–strain curve ($s-e$) with equations (3.7) and (3.8). After necking, the cross-sectional area at the neck must be measured to find the true stress and strain.
The Bridgman Correction

The state of stress at the center of a neck is not uniaxial tension. As material in the center of the neck is being stretched in the axial direction, it must contract laterally. This contraction is resisted by the adjacent regions immediately above and below that have larger cross-sections and are therefore not deforming. The net effect is that the center of a neck is under triaxial tension. Figure 3.10 shows the stress distribution calculated by Bridgman.*

Only that part of the axial stress, which exceeds the lateral stress, is effective in causing yielding. Bridgman showed that the effective part of the stress, $\bar{\sigma}$, is

$$\frac{\bar{\sigma}}{\sigma} = 1/[(1 + 2R/a) \ln[1 + a/(2R)]],$$

(3.10)

where $\sigma$ is the measured stress, $F/A$; $a$ is the radius of specimen at the base of the neck; and $R$ is the radius of curvature of the neck profile.

Figure 3.11 is a plot of the Bridgman correction factor, $\sigma/\sigma$, as a function of $a/R$ according to equation (3.10). A simple way of measuring the radius of curvature, $R$, can be measured by sliding a calibrated cone along the neck until it becomes tangent at the base of the neck.

Temperature Rise

Most of the mechanical energy expended by the tensile machine is liberated as heat in the tensile specimen. If the testing is rapid, little heat is lost to the surrounding, and the temperature rise can be surprisingly high. With very slow testing, most of the heat is dissipated to the surroundings so the temperature rise is much less.

**EXAMPLE PROBLEM 3.2:** Calculate the temperature rise in a tension test of a low-carbon steel after a tensile elongation of 22%. Assume that 95% of the energy goes to heat and remains in the specimen. Also assume that Figure 3.3

![Figure 3.10. Stress distribution across a neck (left) and corresponding geometry of the neck (right). Both axial and lateral tensile stresses are a maximum at the center of the neck.](image)

represents the stress–strain curve of the material. For steel, the heat capacity is 447 J/kgK, and the density is 7.88 Mg/m³.

**Solution:** The heat released equals 0.95∫sde. From Figure 3.3, ∫sde = savel = about 34 × 0.22 ksi = 51.6 MPa = 61.6 MJ/m³. ∆T = Q/C, where Q = 0.95 (61.6 × 10⁶ J/m³), and C is heat capacity per volume = (7.8 × 10³ kg/m³) (447 J/kg/K). Substituting, ∆T = 0.95(61.6 × 10⁶ J/m³)/[(7.8 × 10³ kg/m³) (447 J/kg/K)] = 17°C. This is a moderate temperature rise. For high-strength materials, the rise can be much higher.

**Sheet Anisotropy**

The angular variation of yield strength in many sheet materials is not large. However, such a lack of variation does not indicate that the material is isotropic. The parameter that is commonly used to characterize the anisotropy is the *strain ratio* or *R value* (Figure 3.12). This is defined as the ratio, R, of the contractile strain in the width direction to that in the thickness direction during a tension test,

\[ R = \frac{\varepsilon_w}{\varepsilon_t}. \] (3.11)

If a material is isotropic, the width and thickness strains, εw and ε1, are equal, so for an isotropic material, R = 1. For most materials, however, R is usually either greater or less than 1 in real sheet materials. Direct measurement of the thickness strain in thin sheets is inaccurate. Instead, ε1 is usually deduced from the width and length strains, εw and ε1 assuming constancy of volume, ε1 = −εw − ε1, .

\[ R = -\frac{\varepsilon_w}{(\varepsilon_w + \varepsilon_1)}. \] (3.12)

To avoid constraint from the shoulders, strains should be measured well away from the ends of the gauge section. Some workers suggest that the strains be

* Some authors use the symbol, r, instead of R.
measured when the total elongation is 15%, if this is less than the necking strain. The change of $R$ during a tensile test is usually quite small, and the lateral strains at 15% elongation are great enough to be measured with accuracy.

**EXAMPLE PROBLEM 3.3:** Consider the accuracy of $R$ measurement for a material having an $R$ of about 1.00. Assume the width is about 0.5 in. and can be measured to an accuracy of ±0.001 in., which corresponds to an uncertainty of strain of ±0.001/0.5 = ±0.002. The errors in measuring the length strain are much smaller. Estimate the error in finding $R$ if the total strain was 5%. How would this be reduced if the total strain was 15%?

**Solution:** At an elongation of 5%, the lateral strain would be about 0.025, so the error in $\varepsilon_w$ would be ±0.002/0.025 = 8%. If the $R$ value were 1, this would cause an error in $R$ of approximately $0.08 \times 0.5/(1 - 0.5)$ or 8%. At 15% elongation, the percent error should be about one third of this or roughly ±3%.

The value of $R$ usually depends on the direction of testing. An average $R$ value is conventionally taken as

$$\bar{R} = (R_0 + R_{90} + 2R_{45})/4.$$  (3.13)

The angular variation of $R$ is characterized by $\Delta R$, defined as

$$\Delta R = (R_0 + R_{90} - 2R_{45})/2.$$  (3.14)

Both are important in analyzing what happens during sheet metal forming.

**Measurement of Force and Strain**

In most tensile testing machines, the force is applied through load cells. The load cell is built so that it will deform elastically under the applied loads. The amount of elastic deformation is sensed and converted to an electric signal that in turn may be recorded electronically or used to drive a recording pen.

Several methods may be used to measure strain. With ductile polymers and metals, the deformation is often so great that it may be calculated from the
cross-head movement. In this case, no direct measurements on the specimen are required. With screw-driven machines, the cross-head movement can be deduced from time. If it is assumed that all the cross-head displacement corresponds to specimen elongation, the engineering strain is simply the cross-head displacement divided by the gauge length. This procedure ignores the fact that some of the cross-head movement corresponds to elastic distortion of the machine and the gripping system as well as some plastic deformation in the shoulders of the specimen. The machine displacements can be found as a function of load by running a calibration experiment with a known specimen. Subtracting these machine grip displacements from the cross-head displacement will increase the accuracy of the method.

For small strains, much greater accuracy can be achieved with an extensometer attached to the specimen. An extensometer is a device specifically designed to measure small displacements by using resistance strain gauges, differential capacitances, or differential inductances. The change in resistance, capacitance, or inductance is converted to an electric signal that in turn controls a pen or paper drive or is stored in a computer file.

**Axial Alignment**

For measurement of strains in the elastic region, substantial errors may result with the use of a single extensometer unless the specimen is very straight and axially aligned. Nonaxial alignment and specimen curvature have analogous effects during loading, causing a bending or unbending of the specimen, as sketched in Figure 3.13. The extensometer will respond to the bending as well as the axial elongation. A simple way of compensating for bending and misalignment is to mount two extensometers on opposite sides of the specimen. The averaged response will not be affected by the bending.

Axiality of loading is of particular importance during the testing of brittle materials. In bending, the load on one side of the specimen is higher than on the opposite side, but the average stress deduced from the load will be less than the stress on the more heavily stressed side. Thus, the recorded tensile strength will be lower than the actual stress at the location where the fracture initiates. With low-carbon steels, nonaxial loading may mask the upper yield point. However, with ductile materials, nonaxiality is not a problem after the initial yielding because the specimens straighten during the first plastic deformation.

![Figure 3.13](image.png) Nonaxiality can result from off-center loading (a). With the bending caused by off-center loading, strain measurements on the outside of the bend will be higher than the centerline strain. Straightening of an initially bent specimen (b) will cause measurements of strain on the outside of the bend to be lower than the strain at the centerline. From W. F. Hosford, *Ibid.*
Special Problems

In testing of wire or rope, there is no simple way of using a reduced gauge section. Instead, the wire or rope is wound around a drum so that friction provides the gripping. When specimens are tested at high or low temperatures, the entire gauge section should be at the test temperature. For low temperature testing, the specimen can be immersed in a constant-temperature liquid bath. For testing at elevated temperatures, the specimen is usually surrounded by a controlled furnace.

Compression Test

Because necking limits the uniform elongation in tension, tension tests are not useful for studying the plastic stress–strain relationships at high strains. Much higher strains can be reached in compression, torsion, and bulge tests. The results from these tests can be used, together with the theory of plasticity (Chapter 5), to predict the stress–strain behavior under other forms of loading.

Much higher strains are achievable in compression tests than in tensile tests. However, two problems limit the usefulness of compression tests: friction and buckling. Friction on the ends of the specimen tends to suppress the lateral spreading of material near the ends (Figure 3.14). A cone-shaped region of dead metal (undeforming material) can form at each end, with the result that the specimen becomes barrel shaped. Friction can be reduced by lubrication, and the effect of friction can be lessened by increasing the height-to-diameter ratio, \( h/d \), of the specimen.

If the coefficient of friction, \( \mu \), between the specimen and platens is constant, the average pressure to cause deformation is

\[
P_{\text{av}} = Y \left( 1 + \frac{\mu d}{h} \right) / 3 + \frac{\left( \frac{\mu d}{h} \right)^2}{12} + \cdots ,
\]

where \( Y \) is the true flow stress of the material. If, however, there is a constant shear stress at the interface, such as would be obtained by inserting a thin film of a soft material (e.g., lead, polyethylene, or Teflon), the average pressure is

\[
P_{\text{av}} = Y + (1/3) k(d/h),
\]

Figure 3.14. Unless the ends of a compression specimen are well lubricated, there will be a conical region of undeforming material (dead metal) at each end of the specimen. As a consequence, the midsection will bulge out or barrel.
where $k$ is the shear strength of the soft material. However, these equations usually do not accurately describe the effect of friction because neither the coefficient of friction nor the interface shear stress is constant. Friction is usually highest at the edges where liquid lubricants are lost, and thin films may be cut during the test by sharp edges of the specimens. Severe barreling caused by friction may cause the sidewalls to fold up and become part of the ends, as shown in Figure 3.15. Periodic unloading to replace or relubricate the film will help reduce these effects.

Although increasing $h/d$ reduces the effect of friction, the specimen will buckle if it is too long and slender. Buckling is likely if the height-to-diameter ratio is greater than about 3. If the test is so well lubricated that the ends of the specimen can slide relative to the platens, buckling can occur for $h/d \geq 1.5$ (Figure 3.16).

One way to circumvent the effects of friction is to test specimens with different diameter/height ratios. The strains at several levels of stress are plotted against $d/h$. By the extrapolating the stresses to $d/h = 0$, the stress levels can be found for an infinitely long specimen in which the friction effects would be negligible (Figure 3.17).

During compression, the load-carrying cross-sectional area increases. Therefore, in contrast to the tension test, the absolute value of engineering stress is greater than the true stress (Figure 3.18). The area increase, together with work hardening, can lead to very high forces during compression tests, unless the specimens are very small.

The shape of the engineering stress–strain curve in compression can be predicted from the true stress–strain curve in tension, assuming that absolute values of true stress in tension and compression are the same at the same absolute strain.

Figure 3.15. Photograph of the end of a compression specimen. The darker central region was the original end. The lighter region outside was originally part of the cylindrical wall that folded up with the severe barreling. From G. W. Pearsall and W. A. Backofen, *Journal of Engineering for Industry, Trans ASME*, v. 85B (1963), pp. 68–76.

Figure 3.16. Problems with compression testing: (a) friction at the ends prevents spreading, which results in barreling; and (b) buckling of poorly lubricated specimens can occur if the height-to-diameter ratio, $h/d$, exceeds about 3. Without any friction at the ends (c), buckling can occur if $h/d$ is greater than about 1.5.
Figure 3.17. Extrapolation scheme for Eliminating frictional effects in compression testing. Strains at different levels of stress \((\sigma_1, \sigma_2, \sigma_3)\) are plotted for specimens of differing heights. The strain for “frictionless” conditions is obtained by extrapolating \(d/h\) to 0.

Values. Equations (3.7) and (3.8) apply, but it must be remembered that both the stress and strain are negative in compression,

\[
e_{\text{comp}} = \exp(\varepsilon) - 1,
\]

and

\[
s_{\text{comp}} = \sigma / (1 + e).
\]

**EXAMPLE PROBLEM 3.3:** In a tensile test, the engineering stress \(s = 100\) MPa at an engineering strain of \(e = 0.20\). Find the corresponding values of \(\sigma\) and \(\varepsilon\). At what engineering stress and strain in compression would the values of \(|\sigma|\) and \(|\varepsilon|\) equal those values of \(\sigma\) and \(\varepsilon\)?

**Solution:** In the tensile test, \(\sigma = s(1 + e) = 100(1.2) = 120\) MPa, \(\varepsilon = \ln(1 + e) = \ln(1.2) = 0.182\). At a true strain of \(-0.182\) in compression, the engineering strain would be \(e_{\text{comp}} = \exp(-0.18) - 1 = -0.1667\), and the engineering stress would be \(s_{\text{comp}} = \sigma / (1 + e) = -120\) MPa / (1 – 0.1667) = -144 MPa.

Figure 3.18. Stress–strain relations in compression for a ductile material. Each point, \(\sigma, \varepsilon, e\), on the true stress–true strain curve corresponds to a point, \(s, e\), on the engineering stress–strain curve. The arrows connect these points.
Compression failures of brittle materials occur by shear fractures on planes at 45 degrees to the compression axis. In materials of high ductility, cracks may occur on the barreled surface, either at 45 degrees to the compression axis or perpendicular to the hoop direction. In the latter case, secondary tensile stresses are responsible. These occur because the frictional constraint on the ends causes the sidewalls to bow outward. Because of this barreling, the axial compressive stress in the bowed walls is lower than in the center. Therefore, a hoop direction tension must develop to aid in the circumferential expansion.

**Plane–Strain Compression**

There are two simple ways of making plane–strain compression tests. Small samples can be compressed in a channel that prevents spreading (Figure 3.19a). In this case, there is friction on the sidewalls of the channel as well as on the platens so the effect of friction is even greater than in uniaxial compression. An alternative way of producing plane–strain compression is to use a specimen that is very wide relative to the breadth of the indenter (Figure 3.19b). This eliminates the sidewall friction, but the deformation at and near the edges deviates from plane strain. This departure from plane strain extends inward for a distance approximately equal to the indenter.
width. To minimize this effect, it is recommended that the ratio of the specimen width to indenter width, \( w/b \), be about 8. It is also recommended that the ratio of the indenter width to sheet thickness, \( b/t \), be about 2. Increasing \( b/t \) increases the effect of friction. Both tests simulate the plastic conditions that prevail during flat rolling of sheet and plate. They find their greatest usefulness in exploring the plastic anisotropy of materials.

**Plane–Strain Tension**

Plane–strain can be achieved in tension with specimens having gauge sections that are much wider than they are long. Figure 3.20 shows several possible specimens and specimen gripping arrangements. Such tests avoid the frictional complications of plane–strain compression. However, the regions near the edges lack the constraint necessary to impose plane strain. At the very edge, the stress preventing contraction disappears so the stress state is uniaxial tension. Corrections must be made for departure from plane–strain flow near the edges.

**Biaxial Tension (Hydraulic Bulge Test)**

Much higher strains can be reached in bulge tests than in uniaxial tension tests. This allows evaluation of the stress–strain relationships at high strains. A set-up for bulge testing is sketched in Figure 3.21. A sheet specimen is placed over a circular hole, clamped, and bulged outward by oil pressure acting on one side. Consider a force balance on a small circular element of radius \( \rho \) near the pole when \( \Delta \theta \) is small (Figure 3.22). Using the small angle approximation, the radius of this element be \( \rho \Delta \theta \), where \( \rho \) is the radius of curvature. The stress, \( \sigma \), on this circular region acts
on an area $2\pi \rho \Delta \theta t$ and creates a tangential force equal to $2\pi \sigma \rho \Delta \theta t$. The vertical component of the tangential force is $2\pi \sigma \rho \Delta \theta t$ times $\Delta \theta$, or $2\pi \sigma \rho (\Delta \theta)^2 t$. This is balanced by the pressure, $P$, acting on an area $\pi (\rho \Delta \theta)^2$ and creating an upward force of $P\pi (\rho \Delta \theta)^2$. Equating the vertical forces,

$$\sigma = Pr/(2t).$$

(3.19)

To find the stress, the pressure, radius of curvature, and radial strain must be measured simultaneously. The thickness, $t$, is then deduced from the original thickness, $t_o$, and the radial strain, $\varepsilon_r$,

$$t = t_o \exp(-2\varepsilon_r).$$

(3.20)

EXAMPLE PROBLEM 3.4: Assume that in a hydraulic bulge test, the bulged surface is a portion of a sphere and that at every point on the thickness of the bulged surface the thickness is the same. (Note: This is not strictly true.) Express the radius of curvature, $\rho$, in terms of the die radius, $r$, and the bulged height, $h$. Also express the thickness strain at the dome in terms of $r$ and $h$. See Figure 3.23.

Solution: Using the Pythagorean theorem, $(\rho-h)^2 + r^2 = \rho^2$, so $2\rho h = h^2 + r^2$ or $\rho = (r^2 + h^2)/(2h)$, where $\rho$ is radius of the sphere and $\theta$ is the internal angle.

The area of the curved surface of a spherical segment is $A = 2\pi rh$. The original area of the circle was $A_o = \pi r^2$, so the average thickness strain is $\varepsilon_t \approx \ln[2\pi rh/(\pi r^2)] = \ln(2h/r)$.

Hydrostatic compression superimposed on the state of biaxial tension at the dome of a bulge is equivalent to a state of through-thickness compression, as shown schematically in Figure 3.24. The hydrostatic part of the stress state has no effect on plastic flow. Therefore, the plastic stress–strain behavior of biaxial tension (in the plane of the sheet) and through-thickness compression are equivalent.
**Torsion Test**

Very high strains can be reached in torsion. The specimen shape remains constant so there is no necking instability or barreling. There is no friction on the gauge section. Therefore torsion testing can be used to study plastic stress–strain relations to high strains. In a torsion test, each element of the material deforms in pure shear, as shown in Figure 3.25. The shear strain, \( \gamma \), in an element is given by

\[
\gamma = r \theta / L,
\]

where \( r \) is the radial position of the element, \( \theta \) is the twist angle, and \( L \) is the specimen length. The shear stress, \( \tau \), cannot be measured directly or even determined unequivocally from the torque. This is because the shear stress, \( \tau \), depends on \( \gamma \), which varies with radial position. Therefore, \( \tau \) depends on \( r \). Consider an annular element of radius, \( r \), and width, \( dr \), having an area, \( 2\pi r dr \). The contribution of this element to the total torque, \( T \), is the product of the shear force on it, \( \tau \cdot 2\pi r dr \), times the lever arm, \( r \),

\[
dT = 2\pi \tau r^2 dr \quad \text{and} \quad T = 2\pi \int_0^R \tau r^2 dr.
\]

Equation (3.22) cannot be integrated directly because \( \tau \) depends on \( r \). Integration requires substitution of the stress–strain (\( \tau - \gamma \)) relation. Handbook equations for torque are usually based on assuming elasticity. In this case, \( \tau = G\gamma \). Substituting this and equation (3.21) into equation (3.22),

\[
T = 2\pi \left( \frac{\theta}{L} \right) \int_0^R r^3 dr = \left( \frac{\pi}{2} \right) (\theta / L) Gr^4.
\]

Because \( \tau_{yz} = G\gamma_{yz} \) and \( \gamma_{yz} = r \theta / L \), the shear stress varies linearly with the radial position and can be expressed as \( \tau_{yz} = \tau_s (r/R) \), where \( \tau_s \) is the shear stress at the surface. The value of \( \tau_s \) for elastic deformation can be found from the measured torque by substituting \( \tau_{yz} = \tau_s (r/R) \) into equation (3.23),

\[
T = 2\pi \int_0^r \tau_s (r/R) r^2 dr = \left( \frac{\pi}{2} \right) \tau_s R^3, \quad \text{or} \quad \tau_s = 2T/(\pi R^3).
\]
If the bar is not elastic, Hooke’s law cannot be assumed. The other extreme is when the entire bar is plastic and the material does not work harden. In this case, $\tau$ is a constant.

**EXAMPLE PROBLEM 3.5:** Consider a torsion test in which the twist is great enough that the entire cross-section is plastic. Assume that the shear yield stress, $\tau$, is a constant. Express the torque, $T$, in terms of the bar radius, $R$, and $\tau$.

**Solution:** The shear force on a differential annular element at a radius, $r$, is $\tau \cdot 2\pi r dr$. This causes a differential torque of $dT = r(\tau \cdot 2\pi r dr)$. Integrating,

$$T = 2\pi \tau \int_0^R r^2 dr = (2/3)\pi R^3 \tau.$$  \hfill (3.25)

If the torsion test is being used to determine the stress–strain relationship, the form the stress–strain relationship cannot be assumed so one does not know how the stress varies with radial position. One way around this problem might be to test thin-wall tube in which the variation of stress and strain across the wall would be small enough that the variation of $\tau$ with $r$ could be neglected. In this case, the integral (equation 3.22) could be approximated as

$$T = 2\pi r^2 \Delta r \tau,$$  \hfill (3.26)

where $\Delta r$ is the wall thickness. However, thin-wall tubes tend to buckle and collapse when subjected to torsion. The buckling problem can be circumvented by making separate torsion tests on two bars of slightly different diameter. The difference between the two curves is the torque-twist curve for cylinder whose wall thickness is half of the diameter difference.

The advantage of torsion tests is that very high strains can be reached, even at elevated temperatures. Because of this, torsion tests have been used to simulate the deformation in metal during hot rolling so that the effects of simultaneous hot deformation and recrystallization can be studied. It should be realized that in a torsion test, the material rotates relative to the principal stress axes. Because of this, the strain path in the material is constantly changing.

**Bend Tests**

Bend tests are used chiefly for materials that are very brittle and difficult to machine into tensile bars. In bending, as in torsion, the stress and strain vary with location. The engineering strain, $e$, varies linearly with distance, $z$, from the neutral plane

$$e = z/\rho,$$  \hfill (3.27)

where $\rho$ is the radius of curvature at the neutral plane as shown in Figure 3.26. Consider bending a plate of width $w$. The bending moment, $dM$, caused by the stress on a differential element at $z$ is the force $\sigma w dz$ times the lever arm, $z$, $dM = \sigma w z dz$. The total bending moment is twice the integral of $dM$ from the neutral plane to the outside surface,

$$M = 2 \int_0^{t/2} \sigma w z dz.$$  \hfill (3.28)
To integrate, \( \sigma \) must be expressed in terms of \( z \). If the entire section is elastic, and \( w \ll t \), \( \sigma = eE \) (or if \( w \gg t \), \( \sigma = eE/(1 - \nu^2) \)). Substituting for \( e \), \( \sigma = (z/r)E \). Integrating

\[
M = 2w(E/\rho) \int_0^{t/2} z^2 \, dz,
\]

\[
M = 2w(E/\rho)(t/2)^3/3 = w(E/\rho)t^3/12. \tag{3.29}
\]

The surface stress \( \sigma_s = E(t/2)/r \). Substituting \( r = wEt^3/(12M) \) from equation (3.29),

\[
\sigma_s = 6M/(wt^2). \tag{3.30}
\]

In three-point bending, \( M = FL/4 \), and in four-point bending, \( M = FL/2 \) as shown in Figure 3.27. The relation between the moment and the stress is different if plastic deformation occurs.

**EXAMPLE PROBLEM 3.6:** Consider a four-point bend test on a flat sheet of width, \( w \), of a material for which the stress to cause plastic deformation is a constant, \( Y \). Assume also that the bend is sharp enough so that the entire thickness is plastic. Derive an expression that can be used to determine \( Y \) from the force, \( F \).

**Solution:** The differential force on an element of thickness, \( dz \), at a distance \( z \) from the center is \( Y(w \, dz) \). This force causes a differential moment,
\[ dM = wYz \, dz. \] The net moment is twice the integral of \( dM \) from the center to the surface \((z = t/2)\). Then

\[ M = 2wY(t/2)^2/2 = wt^2Y/4. \]

Substituting \( M = FL/2 \) (equation 3.30 for four-point bending),

\[ Y = 2FL/wt^2. \]

Measurements of fracture strengths of brittle materials are usually characterized by a large amount of scatter because of preexisting flaws, so many duplicate tests are usually required. The fracture stress is taken as the value of the surface stress, \( \sigma_s \), at fracture. This assumes that no plastic deformation has occurred. See Chapters 14 and 19.

**Hardness Tests**

Hardness tests are simple to make, and they can be made on production parts as quality control checks without destroying the part. They depend on measuring the amount of deformation caused when a hard indenter is pressed into the surface with a fixed force. The disadvantage is that although hardness of a material depends on the plastic properties, the stress–strain relation cannot be obtained. Figure 3.28 shows the indenters used for various tests. The Rockwell tests involve measuring the depth of indentation. There are several different Rockwell scales, each of which uses different shapes and sizes of indenters and different loads. Conversion from one scale to another is approximate and empirical.

<table>
<thead>
<tr>
<th>Test</th>
<th>Indenter</th>
<th>Side View</th>
<th>Top View</th>
<th>Load</th>
<th>Formula for Hardness Number</th>
</tr>
</thead>
<tbody>
<tr>
<td>Brinell</td>
<td>10-mm sphere of steel or tungsten carbide</td>
<td></td>
<td></td>
<td>( P )</td>
<td>BHN = ( \frac{2P}{\pi(D - \sqrt{D^2 - d^2})} )</td>
</tr>
<tr>
<td>Vickers</td>
<td>Diamond pyramid</td>
<td></td>
<td></td>
<td>( P )</td>
<td>VHN = 1.72 ( P/d_1 ^2 )</td>
</tr>
<tr>
<td>Knoop microhardness</td>
<td>Diamond pyramid</td>
<td></td>
<td></td>
<td>( P )</td>
<td>KHN = 14.2 ( P/l^2 )</td>
</tr>
<tr>
<td>Rockwell A)</td>
<td>Diamond cone</td>
<td></td>
<td></td>
<td>60 kg</td>
<td>( R_A = )</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>150 kg</td>
<td>( R_C = ) 100 – 500 t</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>100 kg</td>
<td>( R_p = )</td>
</tr>
<tr>
<td>Rockwell B)</td>
<td>½-in.-diameter steel sphere</td>
<td></td>
<td></td>
<td>100 kg</td>
<td>( R_B = )</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>60 kg</td>
<td>( R_F = ) 130 – 500 t</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>150 kg</td>
<td>( R_G = )</td>
</tr>
<tr>
<td>Rockwell C)</td>
<td>½-in.-diameter steel sphere</td>
<td></td>
<td></td>
<td>100 kg</td>
<td>( R_C = )</td>
</tr>
</tbody>
</table>

Brinell hardness is determined by pressing a ball, 1 cm in diameter, into the surface under a fixed load (500 or 3,000 kg). The diameter of the impression is measured with an eyepiece and converted to hardness. The scale is such that the Brinell hardness number, $H_B$, is given by

$$H_B = \frac{F}{A_s}$$

(3.31)

where $F$ is the force expressed in kg and $A_s$ is the spherical surface area of the impression in mm$^2$. This area can be calculated from

$$A_s = \pi D \left[ D - (D^2 - d^2)^{1/2} \right]/2$$

(3.32)

where $D$ is the ball diameter and $d$ is the diameter of the impression, but it is more commonly found from tables.

Brinell data can be used to determine the Meyer hardness, $H_M$, which is defined as the force divided by the projected area of the indentation, $\pi d^2/4$. The Meyer hardness has greater fundamental significance because it is relatively insensitive to load. Vickers and Knoop hardnesses are also defined as the indentation load divided by the projected area. They are nearly equal for the same material. Indentations under different loads are geometrically similar unless the indentation is so shallow that the indenter tip radius is not negligible compared to the indentation size. For this reason, the hardness does not depend on the load. For Brinell, Knoop, and Vickers hardness, a rule of thumb is that the hardness is about three times the yield strength when expressed in the same units. (Note: Hardness is conventionally expressed in kg/mm$^2$ and strength in MPa.) To convert kg/mm$^2$ to MPa, multiply by 9.807. Figure 3.29 shows approximate conversions between several hardness scales.

![Figure 3.29](image_url)
The Brinell, Meyers, Vickers, and Knoop hardness numbers are defined as the force on the indenter divided by area. (This is the projected area, except in the case of the Brinell number.) The hardness number is related to the yield strength measured at the strain characteristic of the indentation. For a nonwork hardening material under plane–strain indentation, theoretical analysis predicts

\[ H = \frac{2}{\sqrt{3}} \left(1 + \frac{\pi}{2}\right) Y = 2.97 Y \approx 3Y. \]  

(3.33)

There are several approximate relations between the different scales.

\[ B \approx 0.95 V, \]  

(3.34)

\[ K \approx 1.05 V, \]  

(3.35)

\[ R_C \approx 100 - 1,480/\sqrt{B}, \]  

(3.36)

and

\[ R_B \approx 134 - 6,700/B, \]  

(3.37)

where \( B, V, K, R_C, \) and \( R_B \) are the Brinell, Vickers, Knoop, Rockwell C, and Rockwell B hardness numbers. Another useful approximation for steels is that the tensile strength (expressed in psi) is about 500 times the Brinell hardness number (expressed in kg/mm\(^2\)).

Possible errors in making hardness tests include:

1. Making indentations too close to an edge of the material. If the plastic zone around an indenter extends to an edge, the reading will be too low.
2. Making an indentation too close to a prior indentation. If the plastic zone around an indenter overlaps that of a prior indentation, the strain hardening during the prior test will cause the new reading to be too high.
3. Making too large an indentation on a thin specimen. If the plastic zone penetrates to the bottom surface, the reading will be in error.

With Rockwell tests, it is important that the bottom of the specimen be flat and supported on a rigid anvil. This is necessary because the dial reading, which indicates the hardness by measuring the depth of penetration, is influenced by bending or settling of the specimen, with the result of recording too low a hardness.

Among the other hardness tests is the Shore scleroscope test, which measures the rebound of a ball dropped from a fixed height onto the surface to be measured. This works on the principle that the energy absorbed when the ball hits the surface is a measure of hardness. With hard materials, little energy is absorbed, so the rebounds are high.

Mineralogists frequently classify minerals by the Moh’s scratch hardness. The system is based on ranking minerals on a scale of 1 to 10. The scale is such that a mineral higher on the scale will scratch a mineral lower on the scale. The scale is arbitrary and not well suited to metals because most metals tend to fall in the range between 4 and 8. Actual values vary somewhat with how the test is made (e.g., the angle of inclination of the scratching edge). Figure 3.30 shows the approximate relationship between Vickers and Moh’s hardesses.
Mutual Indentation Hardness

Hardness tests are normally made with indenters that do not deform. However, it is possible to measure hardness when the indenters do deform. This is useful at very high temperatures, where it is impossible to find a suitable material for the indenter. Defining the hardness as the indentation force/area of indentation, the hardness, $H$, of a material is proportional to its yield strength,

$$H = cY,$$  \hspace{1cm} (3.38)

where $c$ depends mainly on the geometry of the test. For example, in a Brinell test where the indenting ball is much harder than the test material, $c \approx 2.8$. For mutual indentation of crossed wedges, $c \approx 3.4$ (Figure 3.31) and for crossed cylinders, $c \approx 2.4$. The constant $c$ has been determined for other geometries and for cases where the indenters have different hardnesses. The forces in automobile accidents have been estimated from examining the depths of mutual indentations.

Figure 3.30. Relation between Vickers and Moh’s hardness scales.

Figure 3.31. Mutual indentation of two wedges. The hardness is the force/area.
Figure 3.32. Leonardo’s sketch of a system for tensile testing wire. The load was applied by pouring sand into a basket suspended from the wire.

REFERENCES


Notes

Leonardo da Vinci (1452–1519) described a tensile test of a wire. Fine sand was fed through a small hole into a basket attached to the lower end of the wire until the wire broke. Figure 3.32 shows his sketch of the apparatus.

Galileo (1564–1642) stated that the strength of a bar in tension was proportional to its cross-sectional area and independent of its length. Petrus von Musschenbrök (1692–1761) devised the tensile testing machine and grip system illustrated in Figure 3.33.

Percy W. Bridgman (1882–1961) was born in Cambridge, Massachusetts, and attended Harvard, where he graduated in 1904 and received his PhD in 1908. He discovered the high-pressure forms of ice, is reputed to have discovered “dry ice,” and wrote a classic book, *Physics of High Pressures*, in 1931. In 1946, he was awarded the Nobel prize in physics for his work at high pressure. Another book, *Studies in Large Plastic Flow and Fracture* (1952), summarizes his work with the mechanics of solids.

The yield point effect in linear polymers may be experienced by pulling the piece of plastic sheet that holds a six-pack of carbonated beverage cans together.

Figure 3.33. Musschenbrök’s design (1729) of a tensile testing machine that uses a lever to increase the applied force. His system for gripping the specimen is shown at the right.
When one pulls hard enough, the plastic will yield and the force drop. A small thinned region develops. As the force is continued, this region will grow. The yield point effect in low-carbon steel may be experienced by bending a low-carbon steel wire. (Florist’s wire works well.) First, heat the wire in a flame to anneal it. Then bend a 6-in. length by holding it only at the ends. Instead of bending uniformly, the deformation localizes to form several sharp kinks. Why? Bend an annealed copper wire for comparison.

An apparatus suggested by Galileo for bending tests is illustrated in Figure 3.34.

**Problems**

1. The results of a tensile test on a steel test bar are given. The initial gauge length was 25.0 mm, and the initial diameter was 5.00 mm. The diameter at the fracture was 2.6 mm. The engineering strain and engineering stress in MPa are

<table>
<thead>
<tr>
<th>Strain</th>
<th>Stress</th>
<th>Strain</th>
<th>Stress</th>
<th>Strain</th>
<th>Stress</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>0.0</td>
<td>0.06</td>
<td>319.8</td>
<td>0.32</td>
<td>388.4</td>
</tr>
<tr>
<td>0.0002</td>
<td>42.0</td>
<td>0.08</td>
<td>337.9</td>
<td>0.34</td>
<td>388.0</td>
</tr>
<tr>
<td>0.0004</td>
<td>83.0</td>
<td>0.10</td>
<td>351.1</td>
<td>0.38</td>
<td>386.5</td>
</tr>
<tr>
<td>0.0006</td>
<td>125.0</td>
<td>0.15</td>
<td>371.7</td>
<td>0.40</td>
<td>384.5</td>
</tr>
<tr>
<td>0.0015</td>
<td>155.0</td>
<td>0.20</td>
<td>382.2</td>
<td>0.42</td>
<td>382.5</td>
</tr>
<tr>
<td>0.005</td>
<td>185.0</td>
<td>0.22</td>
<td>384.7</td>
<td>0.44</td>
<td>378.0</td>
</tr>
<tr>
<td>0.02</td>
<td>249.7</td>
<td>0.24</td>
<td>386.4</td>
<td>0.46</td>
<td>362.0</td>
</tr>
<tr>
<td>0.03</td>
<td>274.9</td>
<td>0.26</td>
<td>387.6</td>
<td>0.47</td>
<td>250.0</td>
</tr>
<tr>
<td>0.04</td>
<td>293.5</td>
<td>0.28</td>
<td>388.3</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.05</td>
<td>308.0</td>
<td>0.30</td>
<td>388.9</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

A. Plot the engineering stress–strain curve.

B. Determine (i) Young’s modulus, (ii) the 0.2% offset yield strength, (iii) the tensile strength, (iv) the percent elongation, and (iv) the percent reduction of area.
2. Construct the true stress–true strain curve for the material in problem 1. Note that necking starts at maximum load, so the construction should be stopped at this point.

3. Determine the engineering strains, \( e \), and the true strains, \( \varepsilon \), for each of the following:
   - A. Extension from \( L = 1.0 \) to \( L = 1.1 \).
   - B. Compression from \( h = 1 \) to \( h = 0.9 \).
   - C. Extension from \( L = 1 \) to \( L = 2 \).
   - D. Compression from \( h = 1 \) to \( h = 1/2 \).

4. The ASM *Metals Handbook* (Vol. 1, 8th ed., p. 1008) gives the percent elongation in a 2-in. gauge section for annealed electrolytic tough-pitch copper as
   - 55% for a 0.505-in.-diameter bar
   - 45% for a 0.030-in.-thick sheet
   - 38.5% for a 0.010-in.-diameter wire.

   Suggest a reason for the differences.

5. The tensile strength of iron-carbon alloys increases as the percent carbon increases up to contents of about 1.5% to 2%. Above this, the tensile strength drops rapidly with increased percent carbon. Speculate about the nature of this abrupt change.

6. The area under an engineering stress–strain curve up to fracture is the energy/volume. The area under a true stress–strain curve up to fracture is also the energy/volume. If the specimen necks, these two areas are not equal. What is the difference? Explain.

7. Suppose it is impossible to use an extensometer on the gauge section of a test specimen. Instead a button head specimen (Figure 3.2c) is used, and the strain is computed from the cross-head movement. There are two possible sources of error with this procedure. One is that the gripping system may deform elastically, and the other is that the button head may be drawn partly through the collar. How would each error affect the calculated true stress and true strain?

8. Equation (3.10) relates the average axial stress in the neck, \( \sigma_z \), to the effective stress, \( \bar{\sigma} \). The variation of the local stresses with distance, \( r \), from the center is given by

   \[
   \sigma_z = \bar{\sigma}(1 + \ln[1 + a/(2R) - r^2/(2aR)]), \quad \text{with} \quad \sigma_x = \sigma_y = \sigma_z - \bar{\sigma}.
   \]

   Derive an expression for the level of hydrostatic stress, \( \sigma_H = (\sigma_x + \sigma_y + \sigma_z)/3 \), at the center in terms of \( a, R, \) and \( \bar{\sigma} \).

9. The tensile strengths of brazed joints between two pieces of steel are often considerably higher than the tensile strength of the braze material itself. Furthermore, the strengths of thin joints are higher than those of thick joints. Explain.

10. Two strain gauges were mounted on opposite sides of a tensile specimen. Strains measured as the bar was pulled in tension were used to compute Young’s modulus. Readings from one gauge gave a modulus much higher than those from the other gauge. What was the probable cause of this discrepancy?
11. Engineering stress–strain curve from a tension test on a low-carbon steel is plotted in Figure 3.35. From this construct, the engineering stress–strain curve in compression, neglecting friction.

![Figure 3.35. Engineering stress–strain curve for a low-carbon steel](image)

12. Discuss the how friction and inhomogeneous deformation affect the results from
   A. The two types of plane–strain compression tests illustrated in Figure 3.19.
   B. The plane–strain tension tests illustrated in Figure 3.20.

13. Sketch the three-dimensional Mohr’s stress and strain diagrams for a plane–strain compression test.

14. Draw a Mohr’s circle diagram for the surface stresses in a torsion test, showing all three principal stresses. At what angle to the axis of the bar are the tensile stresses the greatest?

15. For a torsion test, derive equations relating the angle, $\psi$, between the axis of the largest principal stress and the axial direction and the angle, $\psi$, between the axis of the largest principal strain and the axial direction in terms of $L$, $r$, and the twist angle, $\theta$. Note that for finite strains, these two angles are not the same.

16. Derive an expression relating the torque, $T$, in a tension test to the shear stress at the surface, $\tau_s$, in terms of the bar diameter, $D$, assuming that the bar is
   A. Entirely elastic so $\tau$ varies linearly with the shear strain, $\gamma$.
   B. Entirely plastic and does not work harden so the shear stress, $\tau$, is constant.

17. The principal strains in a circular bulge test are the thickness strain, $\varepsilon_t$; the circumferential (hoop) strain, $\varepsilon_c$; and the radial strain, $\varepsilon_r$. Describe how the ratio, $\varepsilon_c/\varepsilon_r$, varies over the surface of the bulge. Assume that the sheet is locked at the opening.

18. Derive an expression for the fracture stress, $S_f$, in bending as a function of $F_t$, $L$, $w$, and $t$ for three-point bending of a specimen having a rectangular cross-section, where $F_t$ is the force at fracture, $L$ is the distance between supports, $w$ is the specimen width, and $t$ is the specimen thickness. Assume the deformation is elastic, the
deflection, $y$, in bending is given by $y = \alpha FL^3/(EI)$, where $\alpha$ is a constant and $E$ is Young’s modulus. How would you expect the value of $\alpha$ to depend on the ratio of $t/w$?

19. Equation (3.30) gives the stresses at the surface of bend specimens. The derivation of this equation is based on the assumption of elastic behavior. If there is plastic deformation during the bend test, will the stress predicted by this equation be (a) too low, (b) too high, (c) either too high or too low depending on where the plastic deformation occurs, or (d) correct?

20. By convention, Brinell, Meyer, Vickers, and Knoop hardness numbers are stresses expressed in units of kg/mm$^2$, which is not an SI unit. To what stress, in MPa, does a Vickers hardness of 100 correspond?

21. In making Rockwell hardness tests, it is important that the bottom of the specimen is flat so that the load does not cause any bending of the specimen. On the other hand, this is not important in making Vickers or Brinell hardness tests. Explain.