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The stereographic projection
and its uses

12.1 Introduction

In Chapter 6 we showed how both the orientations and the \( d_{\text{hkl}} \) spacings of planes could be represented in terms of reciprocal lattice vectors and how these vectors could be used to determine the angles between planes and to specify zones and zone axes. We now need a method of representing the planes or faces in a crystal ‘all at once’ so that we can recognize the zones to which they belong and determine the angles between them without the need for repetitive calculations. The ‘crystal drawings’ such as Fig. 5.7 are obviously inadequate in this respect; half of the crystal faces are ‘hidden from view’ and we can only recognize zones with difficulty by looking for the parallel lines of intersections between the (visible) faces. The stereographic projection provides an important method of overcoming these difficulties. It is similar to the reciprocal lattice construction in that in both cases planes or faces are represented by their normals.

The stereographic projection is a very ancient geometrical technique; it originated in the second century A.D. in the work of the Alexandrian astronomer Claudius Ptolemy who used it as a means of representing the stars on the heavenly sphere. The original Greek manuscript is lost, but the work comes down to us in a sixteenth-century Latin translation from an Arabic commentary entitled The Planispherium. The stereographic projection was first applied to crystallography in the work of F. E. Neumann∗ and was further developed by W. H. Miller.∗

The geometry of the stereographic projection may be described very simply. First, the crystal is imagined to be at the centre of a sphere (the stereographic sphere); the normals to the crystal faces are imagined to radiate out from the centre and to intersect the sphere in an array of points. Each point on the sphere therefore represents a crystal face or plane (and is labelled with the appropriate Miller index) just as a ‘point’ on the surface of the Earth represents a town or city. The (angular) distance between two points is equal to the angle between the corresponding planes and is determined in the same way that we find the angular distance between, say, Bangkok and New York: we take our private aeroplane and, uninhibited by Traffic Control, fly in a great circle between the two. Lines of longitude are simply special cases of great circles which pass through the north and south poles, the angular distances between the poles being of course 180°. Lines of latitude are called small circles and represent different angular distances from

∗ Denotes biographical notes available in Appendix 3.
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The north and south poles. The largest small circle is the equator which is also a great circle \(90^\circ\) from the north and south poles.

The second step is to find a method of representing this three-dimensional information in two dimensions. One method is to project all the points on the surface of the sphere by parallel lines on to a flat disc called the plane of projection—in the same way as we see in effect all the features on the surface of the Moon, Fig. 12.1. This has the disadvantage that features near the edge of the Moon are seen foreshortened or 'edge on' and that craters, which we know to be circular, are seen as elliptical in shape. The geometry is shown in Fig. 12.2(a), the point of view being taken, in 'Earth' terms, from above the north pole. Five points \(15^\circ, 30^\circ, 45^\circ, 60^\circ\) and \(75^\circ\) from the north pole are shown and it is easily seen how equal angles are represented by smaller distances as we move out from the centre. However, in the stereographic projection we do not project the points on the surface of the sphere in this way but rather project them to the south pole as shown in Fig. 12.2(b). Now the distortion is, as it were, the other way round, equal angles are represented by larger distances as we move out from the centre. However, the stereographic projection has one very important geometrical advantage and that is that circles on the surface of the sphere appear as circles, not ellipses, in the plane of projection. The geometry is shown
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in Fig. 12.3(a). Here a circle in the northern hemisphere (a small circle) is centred about O; all the points on this small circle are at equal angles from O (just as all the points on a line of latitude are at equal angles from the north and south poles). All the points on the circle are projected, as shown, to the south pole giving a cone—not a cone like an ice-cream cone of circular cross-section but one of elliptical cross-section. However, in any such elliptical cone there are two circular cross-sections inclined at equal angles to the axis of the cone, one of which is the circle on the sphere and the other of which lies in the plane of projection (the proof of this is left as Exercise 12.1). Figure 12.3(b) shows a plan view of the equatorial plane, or the stereographic projection of the small circle. Notice that O', the projection of the centre of the circle is displaced towards the centre of the projection and does not coincide with the point of a drawing-compass. This follows from the fact, pointed out above, that in the projection equal angles are not represented by equal distances.

Figures 12.2 and 12.3 show the construction for points in the northern hemisphere. For points in the southern hemisphere we project instead to the north pole and represent the intersections of the lines with the plane of projection by small unfilled circles instead of dots.

Hence, in summary, and to return to our crystal, the stereographic projection consists of an array of points and circles, called stereographic poles or simply poles representing plane normals in the northern and southern hemispheres respectively and each labelled with the appropriate Miller index of the crystal face they represent.
12.2 Construction of the stereographic projection of a cubic crystal

Figure 12.3 shows a cubic crystal at the centre of the stereographic sphere with \{100\}, \{110\} and \{111\} faces and oriented for easy ‘Earth’ reference with the \( z \)-axis in the
direction of the north pole and the $x$ and $y$ axes in the equatorial plane (the stereographic plane of projection). The normals to the crystal faces are shown and their intersections with the sphere labelled with their \{hkl\} indices. The great circles on the sphere represent zones. For example, planes or faces (00$\bar{1}$), (10$\bar{1}$), (100), (101) and (001) all lie in the [010] zone and their normals lie on a great circle—a line of longitude which passes through the north and south poles. Similarly, planes or faces (010), (111), (101), (1$\bar{1}$1) and (0$\bar{1}$0) all lie in the [101] zone and their normals all lie on a great circle in this case inclined at $45^\circ$ to the north and south poles.

Figure 12.5 just shows the poles of planes in the equator and in the northern hemisphere with the crystal and great circles omitted (to avoid the confusion of too many
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Fig. 12.5. Poles of planes in the northern hemisphere (and in the equatorial plane of projection) and the projection lines to the south pole, the intersections of which in the plane of projection are marked by dots and their corresponding \(\{hkl\}\) indices indicate the position of the stereographic poles of the planes.

lines). The lines of projection to the south pole are shown and their intersections with the stereographic plane of projection marked by dots with their corresponding \(\{hkl\}\) indices. Finally, Fig. 12.6 is a ‘plan’ view of the stereographic projection showing the stereographic poles and again the great circles which project as the arcs of circles passing through them. Now look back to Fig. 12.4. It is very important that you are able to visualize the relationships between the faces in the crystal and their representation as poles in the stereographic projection.

In plotting the exact positions of the poles in Fig. 12.6 it is important to remember that equal angles are not represented by equal distances (except, of course, around the perimeter). For example the (101) plane is 45° between (001) and (100) but projects at a smaller distance to (001) as shown in Fig. 12.2(b). Having plotted the positions of the (101), (011), (0\(\bar{1}1\)), etc. poles at their proper scale, the great circles through them ‘automatically’ give the positions of all the other poles or faces in the crystal—both
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Fig. 12.6. A plan view of the stereographic plane of projection (the stereogram) showing the stereographic poles of the faces of the crystal in Fig. 12.4 and including both those faces ‘hidden from view’ in Fig. 12.4 and also additional faces (such as, for example, (112)) not drawn in Fig. 12.4. Great circles on the sphere project as the arcs of circles in the plane of projection.

Those ‘hidden from view’ such as (111) and also ones not shown in the drawing of the crystal in Fig. 12.4. For example (112) lies at the intersection of the zone through (101) and (011) and also the zone through (111) and (001); i.e. by use of the addition rule (Section 5.6.4) $(101) + (011) = (112)$ and $(111) + (001) = (112)$. Clearly, by drawing more great circles and using the addition rule, planes of higher $\{hkl\}$ values can be located (Exercise 12.2).

Finally, we must now consider the points lying in the southern hemisphere where the projection lines are made to the north pole and their intersections with the plane of projection represented by small unfilled circles. These need not be shown as such in Fig. 12.6 because clearly, for this orientation of the crystal (z-axis or (001) at the centre), a plane such as (101) is coincident with (but ‘underneath’) the plane (101).

12.3 Manipulation of the stereographic projection: use of the Wulff net

We now need a ‘stereographic graph paper’ in order to locate stereographic poles, zones and zone axes. One such is the polar net (Fig. 12.7) which shows great circles like lines

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**Note:** The diagram in the text refers to additional great circles and zones, which are critical for understanding the stereographic projection, as illustrated in the stereogram. The text explains how to locate and interpret the stereographic poles and additional faces, emphasizing the use of the addition rule and the importance of visualizing these projections for crystallographic analysis. The Wulff net is introduced as a tool for manipulating these projections, facilitating the location and identification of planes and zones in crystallographic studies.